

# The Total Green's Function of a Non-Interacting System\*

David Roberts  
Jefferson Physical Laboratory,  
Harvard University, Cambridge, MA 02138

(Dated: February 26, 2016)

Despite its centrality in the mathematical structure of perturbative many-body theory, the total Green's function for the many-body time-dependent Schrodinger equation has been ignored for decades, superseded by single-particle Green's functions, for which a vast portion of the literature has been devoted. In this paper, we give the first computation the total Green's function for the time-dependent Schrodinger equation for a non-interacting system of identical particles, setting the stage for a fresh interpretation of perturbative many-body physics.

## I. INTRODUCTION

Condensed matter physicists use effective field theories all the time. In trying to engineer emergent phenomena into novel materials at low-energy scales, following [2], it would be nice to be able to define a family of effective Hamiltonians

$$\{H^{\text{eff}}[\Lambda]\}_{\Lambda \geq 0}$$

at every energy-scale  $\Lambda$ , for a fixed material, and formulate a renormalization group flow naturally in this context, that can interpolate between between a microscopic and effective many-body theory. As it turns out, the total Green's function

$$G^{\text{eff}}[\Lambda] \equiv (\partial_t + iH^{\text{eff}}[\Lambda])^{-1}$$

is absolutely necessary to implement this program efficiently (See [1]). However, this puts us in an awkward position, because such an object is foreign to the condensed matter literature.

## II. THE TOTAL GREEN'S FUNCTION

Time-evolution of a many-body quantum system is given by the time-dependent many-body Schrodinger equation (in units where  $\hbar = 1$ ),

$$(\partial_t + iH)\Psi = 0.$$

In this paper, we compute the Green's function of the linear differential equation above, which we'll call the **total Green's function** of our many-body system:

$$G \equiv (\partial_t + iH)^{-1}.$$

Despite being such a fundamental mathematical quantity, surprisingly no one has actually computed the total

Green's function for a many-body system.

In the case of a complicated interacting system, this computation is intractable. However, we can give the first computations of this function in the case that the dynamics is non-interacting. We can then compute the total Green's function of a general system via perturbation theory, but we will leave that for another article.

Since our Hamiltonian is non-interacting, it sends each  $k$ -particle portion of the total Hilbert space to itself: so the total Green's function  $G$  also sends each  $k$ -particle portion of its Hilbert space to itself:

$$G = \bigoplus_{k \geq 0} G_k$$

The terms on the right-hand-side are the **k-particle Green's functions**. Therefore, to compute the total Green's function, it will suffice to compute  $G_k$  for all  $k \geq 0$ .

## III. COMPUTATION OF $G_0$ AND $G_1$

**Note:** For conceptual simplicity, throughout this article, we will assume that the single-particle Hilbert space is finite dimensional, with basis  $\{f_i\}$ .

The first two terms in the direct sum have already been computed in the literature, and we will not waste any time and just briefly mention the results here:

$$(G_0)_t^{t'} = (\partial_t + iH_0)^{-1}_t^{t'}$$

Since  $H_0 \equiv 0$  (see [4]), we get the standard theta-function, the integral kernel of the differential operator  $\partial_t$ :

$$(G_0)_t^{t'} = ((\partial_t)^{-1})_t^{t'} = \theta(t - t')$$

The operator  $G_1$  is all over the many-body literature: it is called "the propagator", or sometimes "the single-particle Green's function". It has the following matrix

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\* A footnote to the article title

elements:

$$(G_1)_{it}^{jt'} = \theta(t-t') \langle [a_{\pm}(f_i, t), a_{\pm}^{\dagger}(f_j, t')]_{\pm} \rangle_{T=0}$$

Computation of  $G_1$  is usually given as a trivial exercise in many-body textbooks, such as [3]. Already, with  $k = 0, 1$ , we can see a pattern forming. We will extrapolate to general values of  $k$  in the next section.

#### IV. COMPUTATION OF $G_k$ FOR $k \geq 2$

We now state our main result:

**Theorem IV.1 (The Total Green's Function)** For  $k \geq 2$ , define  $\tilde{G}_k$  by the following matrix elements:

$$(\tilde{G}_k)_{i_1 \dots i_k t}^{j_1 \dots j_k t'} = \theta(t-t') \langle [a_{\pm}(f_{i_1}, t), a_{\pm}^{\dagger}(f_{j_1}, t')]_{\pm} \dots [a_{\pm}(f_{i_k}, t), a_{\pm}^{\dagger}(f_{j_k}, t')]_{\pm} \rangle_{T=0}$$

Then the  $k$ -particle Green's function  $G_k$  of the non-interacting system is simply the restriction of  $\tilde{G}_k$  to the appropriate (anti-)symmetric subspace.

*Proof.* For a system of non-interacting identical particles, there exists a basis of the single-particle Hilbert space with respect to which the Hamiltonian can be rewritten as

$$\sum_{ij} A_{ij} a_{\pm}^{\dagger}(f_i) a_{\pm}(f_j) = \sum_i B_i a_{\pm}^{\dagger}(g_i) a_{\pm}(g_i).$$

**Lemma:** In the associated basis of our un-(anti)-symmetrized Fock-space, the matrix elements of  $\tilde{G}_k$  become

$$(\tilde{G}_k)_{i_1 \dots i_k t}^{j_1 \dots j_k t'} = \theta(t-t') \langle [a_{\pm}(g_{i_1}, t), a_{\pm}^{\dagger}(g_{j_1}, t')]_{\pm} \dots [a_{\pm}(g_{i_k}, t), a_{\pm}^{\dagger}(g_{j_k}, t')]_{\pm} \rangle_{T=0}$$

*Proof of Lemma.* Let the change-of-basis be expressed as follows:

$$g_i = U_i^j f_j$$

We now use the bilinearity of the (anti-)commutator, and the (anti-)linearity of  $(a_{\pm}), a_{\pm}^{\dagger}$ :

$$\begin{aligned} (\tilde{G}_k)_{i_1 \dots i_k t}^{j_1 \dots j_k t'} &= \theta(t-t') \langle [a_{\pm}(g_{i_1}, t), a_{\pm}^{\dagger}(g_{j_1}, t')]_{\pm} \dots [a_{\pm}(g_{i_k}, t), a_{\pm}^{\dagger}(g_{j_k}, t')]_{\pm} \rangle_{T=0} \\ &= \theta(t-t') \langle U_{i_1}^{l_1} (U_{j_1}^{m_1})^* [a_{\pm}(f_{l_1}, t), a_{\pm}^{\dagger}(f_{m_1}, t')]_{\pm} \dots U_{i_k}^{l_k} (U_{j_k}^{m_k})^* [a_{\pm}(f_{l_k}, t), a_{\pm}^{\dagger}(f_{m_k}, t')]_{\pm} \rangle_{T=0} \\ &= \theta(t-t') U_{i_1}^{l_1} (U_{j_1}^{m_1})^* \dots U_{i_1}^{l_1} (U_{j_1}^{m_1})^* (\tilde{G}_k)_{l_1 \dots l_k t}^{m_1 \dots m_k t'} \end{aligned}$$

Since  $U_i^j = (f_i, U f_j)$  is unitary, we can write the above as follows:

$$(\tilde{G}_k)_{i_1 \dots i_k t}^{j_1 \dots j_k t'} = (U^{\dagger})_{m_k}^{j_k} \dots (U^{\dagger})_{m_1}^{j_1} (\tilde{G}_k)_{l_1 \dots l_k t}^{m_1 \dots m_k t'} U_{i_1}^{l_1} \dots U_{i_k}^{l_k}$$

Which satisfies the formula for induced change-of-basis on the un-(anti)-symmetrized Fock space  $\square$ .

Now we resume our proof. Since the Hamiltonian is diagonal in the basis  $\{g_i\}$ , it may be verified that

$$a_{\pm}^{\dagger}(g_j, t') = e^{-i(t-t')B_j} a_{\pm}^{\dagger}(g_j, t)$$

Therefore, we can begin to simplify the matrix elements of  $\tilde{G}_k$  as follows:

$$(\tilde{G}_k)_{i_1 \dots i_k t}^{j_1 \dots j_k t'} = \theta(t-t') \langle e^{-i(t-t')B_{j_1}} [a_{\pm}(g_{i_1}, t), a_{\pm}^{\dagger}(g_{j_1}, t)]_{\pm} \dots e^{-i(t-t')B_{j_k}} [a_{\pm}(g_{i_k}, t), a_{\pm}^{\dagger}(g_{j_k}, t)]_{\pm} \rangle_{T=0}$$

Using the equal-time (anti-)commutation relations

$$\{a_{\pm}(f, t), a_{\pm}(g, t)\} = (f, g),$$

we get

$$(\tilde{G}_k)_{i_1 \dots i_k t}^{j_1 \dots j_k t'} = \theta(t-t') \delta_{i_1 \dots i_k}^{j_1 \dots j_k} \cdot e^{-i(t-t')(B_{j_1} + \dots + B_{j_k})}.$$

We now compute the time-derivative of the above expression:

$$\begin{aligned} (\partial_t \tilde{G}_k)_{i_1 \dots i_k t}^{j_1 \dots j_k t'} &= \delta(t-t') \delta_{i_1 \dots i_k}^{j_1 \dots j_k} \cdot e^{-i(t-t')(B_{j_1} + \dots + B_{j_k})} \\ &\quad - i(B_{j_1} + \dots + B_{j_k}) (\tilde{G}_k)_{i_1 \dots i_k t}^{j_1 \dots j_k t'} \end{aligned}$$

Since the first term in the above expression vanishes for  $t \neq t'$ , we can eliminate the phase-factor which multiplies it, yielding

$$\begin{aligned} (\partial_t \tilde{G}_k)_{i_1 \dots i_k t}^{j_1 \dots j_k t'} &= \delta(t-t') \delta_{i_1 \dots i_k}^{j_1 \dots j_k} \\ &\quad - i(B_{j_1} + \dots + B_{j_k}) (\tilde{G}_k)_{i_1 \dots i_k t}^{j_1 \dots j_k t'}. \end{aligned}$$

Recall that we can extend a non-interacting Hamiltonian to the un-(anti)-symmetrized Fock space by letting

$$\tilde{H}_k(f_1 \otimes \dots \otimes f_k) \equiv \sum_i f_1 \otimes \dots \otimes H_1 f_i \otimes \dots \otimes f_k,$$

where  $H_1$  is our associated single-particle Hamiltonian (see the first page for the definition of  $H_1$ ). Therefore, acting on our Green's function with  $i\tilde{H}_k$ , we get

$$(i\tilde{H}_k \tilde{G}_k)_{i_1 \dots i_k t}^{j_1 \dots j_k t'} = i(B_{j_1} + \dots + B_{j_k}) (\tilde{G}_k)_{i_1 \dots i_k t}^{j_1 \dots j_k t'}.$$

Where we get the simple factor because we are implicitly in an eigenbasis of  $\tilde{H}$ , and so the action of  $\tilde{H}$  is diagonal. Therefore, putting it all together, we get

$$((\partial_t + i\tilde{H}_k) \circ (\tilde{G}_k))_{i_1 \dots i_k t}^{j_1 \dots j_k t'} = \delta(t-t') \delta_{i_1 \dots i_k}^{j_1 \dots j_k}$$

In basis-independent language, this is the simple identity  $(\partial_t + i\tilde{H}_k) \circ \tilde{G}_k = I$ , i.e., we have verified that, on the un-(anti)-symmetrized Fock space,

$$\tilde{G}_k = (\partial_t + i\tilde{H}_k)^{-1}.$$

Therefore, restricting this operator expression to the (anti)-symmetrized Fock space  $\mathcal{F}_{\pm}$  yields our desired identity:

$$G_k = (\partial_t + iH_k)^{-1}. \quad \square$$

## V. CONCLUDING REMARKS

In this paper, we computed the Green's function

$$G = (\partial_t + iH)^{-1}$$

for the time-dependent Schrodinger equation, in the case of non-interacting identical particles, by computing each term in the direct-sum decomposition. The final result was  $G = \bigoplus_{k \geq 0} P_{\pm} \tilde{G}_k P_{\pm}$ , where  $P_{\pm}$  is the (anti)symmetrization operator, and

$$(\tilde{G}_k)_{i_1 \dots i_k}^{j_1 \dots j_k} = \theta(t - t') \left\langle [a_{\pm}(f_{i_1}, t), a_{\pm}^{\dagger}(f_{j_1}, t')]_{\pm} \dots [a_{\pm}(f_{i_k}, t), a_{\pm}^{\dagger}(f_{j_k}, t')]_{\pm} \right\rangle_{T=0}.$$

i.e., the  $k$ -particle Green's function  $G_k$  of the non-interacting system is simply the restriction of  $\tilde{G}_k$  to the appropriate (anti)symmetric subspace. **Example:** for a non-interacting system of identical spinless fermions, and in traditional notation,

$$\begin{aligned} \tilde{G}_0(t, t') &= \theta(t - t') \\ \tilde{G}_1(x, x', t, t') &= \theta(t - t') \left\langle \{\Psi(x, t), \Psi^{\dagger}(x', t')\} \right\rangle_{T=0} \\ G_2(x, x', y, y', t, t') &= \theta(t - t') \left\langle \{\Psi(x, t), \Psi^{\dagger}(x', t')\} \right. \\ &\quad \left. \cdot \{\Psi(y, t), \Psi^{\dagger}(y', t')\} \right\rangle_{T=0} \end{aligned}$$

This work paves the way for a reformulation of perturbation and renormalization theory in terms of the full many-body Green's function.

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- [1] *But there is no Hamiltonian*, Jan Dereziński <http://www.fuw.edu.pl/~derezins/nohamiltonian.pdf>  
 [2] K. Costello, *Renormalization and effective field theory*, (2010). Available at <http://www.math.northwestern.edu/~costello/>.

- [3] Bruus, H., and K. Flensberg, 2004, *Many-Body Quantum Theory in Condensed Matter Physics: An Introduction* (Oxford University Press, Oxford).  
 [4] O. Bratteli and D.W. Robinson. *Operator algebras and quantum statistical mechanics I, II*. Springer, New York, 1979-1981. [There is a 2nd ed. (1997) of vol. II].